MA 182 Trigonometric Substitutions
Section 5.7

Recall that $\dfrac{d(\sin^{-1} x)}{dx} = \dfrac{1}{\sqrt{1-x^2}}$ and $\dfrac{d(\tan^{-1} x)}{dx} = \dfrac{1}{1+x^2}$

Using these basic formulas,

$$\frac{d(\sin^{-1} x/a)}{dx} = \frac{1/a}{\sqrt{1-\dfrac{x^2}{a^2}}} = \frac{1}{\sqrt{a^2-x^2}} \qquad \text{and} \qquad \frac{d(\tan^{-1} x/a)}{dx} = \frac{1/a}{1+\dfrac{x^2}{a^2}} = \frac{a}{a^2+x^2}$$

Based on these differentiation formulas, we have the following integration formulas:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C \qquad \text{and} \qquad \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Examples:

(1) $$\int \frac{1}{\sqrt{9-x^2}}\,dx = \sin^{-1}\frac{x}{3} + C$$

(2) $$\int \frac{1}{x^2+2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

Other integrals which contains expressions of the form $\sqrt{a^2-x^2}$ or $\sqrt{a^2+x^2}$ (a > 0) can often be evaluated by making use of a substitution which involves appropriate inverse trigonometric functions. The basic idea is to make a substitution that will eliminate the radical.

If the integrand contains $\sqrt{a^2-x^2}$, a substitution involving the inverse sine function will often be useful. In this case,

Let $q = \sin^{-1}\dfrac{x}{a}$, $(-p/2 \le q \le p/2)$, so that $\sin q = \dfrac{x}{a}$.

Therefore $x = a\sin q$, $dx = a\cos q \; dq$ and

$$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2 q} = \sqrt{a^2(1-\sin^2 q)} = a\sqrt{\cos^2 q} = a\left|\cos q\right| = a\cos q$$

Note that $\left|\cos q\right| = \cos q$ because $-p/2 \le q \le p/2$ and $\cos q > 0$ in quadrants I and IV.

The following example illustrates the technique.

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx$$

If the integrand contains $\sqrt{a^2+x^2}$, a substitution involving the inverse tangent function will often be useful.

In this case,

Let $\theta = \tan^{-1}\dfrac{x}{a}$, $-\pi/2 < \theta < \pi/2$, so that $\tan\theta = \dfrac{x}{a}$.

Therefore $x = a\tan\theta$, $dx = a\sec^2\theta$, and

$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = a\sqrt{1+\tan^2\theta} = a\sqrt{\sec^2\theta} = a\left|\sec\theta\right| = a\sec\theta$$

Note that $\left|\sec\theta\right| = \sec\theta$ because $-\pi/2 < \theta < \pi/2$ and $\sec\theta > 0$ in quadrants I and IV.

An example follows.

$$\int x^3\sqrt{x^2+9}\,dx$$

Use either a trigonometric or another appropriate substitution to evaluate each indefinite integral.

1. $\int \frac{1}{\sqrt{16-x^2}}\,dx$

2. $\int \frac{x}{\sqrt{9-x^4}}\,dx$ Hint: Let $u = x^2$

3. $\int \frac{x^3}{\sqrt{9-x^4}}\,dx$ Hint: This can be integrated using a trigonometric substitution but there is a much easier way.

4. $\int \sqrt{4-x^2}\,dx$

5. p. 408/#14

6. $\int \frac{1}{x^2+25}\,dx$

7. $\int \frac{x}{x^2+25}\,dx$

8. (a) Verify by differentiation that $\int \sec q\; dq = \ln\left|\sec q + \tan q\right| + C$

   (b) Evaluate the indefinite integral $\int \frac{1}{\sqrt{x^2+4}}\,dx$

9. p. 408/#11 Hint: After using the trigonometric substitution $x = 2\tan q$, rewrite the integral in terms of sines and cosines in order to integrate.

10. p. 408/#12